Question 1. [4 points]

Evaluate the truth value of the following logical expression for p = T, q = F, r = F. Show all steps for credit.

$$(p \vee \neg r) \rightarrow [\, (\neg p \wedge r\,) \rightarrow \neg q\,]$$

$$(T \vee \neg F) \rightarrow [\,(\neg T \wedge F\,) \rightarrow \neg F\,]$$
$$T \rightarrow [(F) \rightarrow T]$$
$$T \rightarrow T$$
$$T$$

Question 2. [12 points, 4 each]

Let p, q and r be the propositions

p: I am playing golf
q: It is snowing
r: It is raining

Write the following statements using the above propositions and logical connectives.

(a) I am playing golf and it is raining.

$$p \wedge r$$

(b) I am playing golf unless it is raining and snowing.

$$\neg(q \wedge r) \rightarrow p$$

(c) It is snowing and I am playing golf, but it is not raining.

$$q \wedge p \wedge \neg r$$

Question 3. [12 points, 4 each]

Let $p(x)$ and $q(x, y)$ be the predicates

$p(x)$: x is in our class,
$q(x, y)$: x knows y,

where the domain of x and y is the set of all students in KFUPM. Write the following statements using symbolic notations:

(a) Someone knows everybody in our class.

$$\exists x\, \forall y\, [(p(y) \rightarrow q(x,y)]$$

(b) Everybody knows somebody in our class.

$$\forall x\, \exists y\, [(p(y) \wedge q(x,y)]$$

(c) Some students in our class do not know each other.

$$\exists x\, \exists y\, [(p(x) \wedge p(y) \wedge \neg q(x,y) \wedge (x \neq y)^{*} \wedge \neg q(y,x)^{*}]$$

***Optional**

Question 4. [10 points, 5 each]

Consider the island of knights and knaves introduced by Smullyan, where knights always tell the truth, and knaves always lie. Determine (without proof) the identity of each individual if they address you in the following ways:

(a) *A* says "Exactly one of us is a knave" and *B* says "We are the same"

A: Knight B: Knave

(b) *A* says "*B* is a knave", *B* says "*A* and *C* are knaves", and *C* says "*A* and *B* are different"

A: Knight B: Knave C: Knight

Question 5. [14 points, 7 each]

(a) Negate the following logical expression so that no negation sign ($\neg$) appears in the expression.

$$\forall x \ \exists y \ \forall z \ [(x > z) \rightarrow (z \neq y)]$$

$$\begin{aligned}
&\neg[\forall x \ \exists y \ \forall z \ [(x > z) \rightarrow (z \neq y)]] \\
&\equiv \exists x \ \forall y \ \exists z \ \neg[(x > z) \rightarrow (z \neq y)]] \\
&\equiv \exists x \ \forall y \ \exists z \ \neg[\neg(x > z) \vee (z \neq y)]] \\
&\equiv \exists x \ \forall y \ \exists z \ [(x > z) \wedge (z = y)]] \text{ by De Morgan's Law}
\end{aligned}$$

(b) Use the logical equivalence laws to show whether the following statement is a *tautology*, *contradiction*, or *contingency*. Clearly <u>show all steps</u> for credit. (Do not use truth tables).

$$((p \vee q) \wedge \neg p) \rightarrow q$$

$$\begin{aligned}
&\equiv \neg[(p \vee q) \wedge \neg p] \vee q && \text{; by the definition of implication} \\
&\equiv [\neg(p \vee q) \vee p] \vee q && \text{; by De Morgan's Law} \\
&\equiv [(\neg p \wedge \neg q) \vee p] \vee q && \text{; by De Morgan's Law} \\
&\equiv [(\neg p \vee p) \wedge (\neg q \vee p)] \vee q && \text{; by Distribution} \\
&\equiv [T \wedge (\neg q \vee p)] \vee q && \text{; by Negation} \\
&\equiv [\neg q \vee p] \vee q && \text{; by Identity Law} \\
&\equiv [\neg q \vee q] \vee p && \text{; Associative/ Commutative Laws} \\
&\equiv T \vee p && \text{; by Negation} \\
&\equiv T && \text{; by Domination Law}
\end{aligned}$$

$\therefore$ It is a tautology

Question 6. [18 points, 9 each]

Consider the following rules of inferences.

Addition	Modus ponens	Hypothetical syllogism	Conjunction
Simplification	Modus tollens	Disjunctive syllogism	Resolution
Universal instantiation/generalization		Existential instantiation/generalization	

(a) Which rule of inference is used in each of the following arguments?

i) Ali is a software engineer. He likes sport. Therefore, some software engineer likes sport.

Existential Generalization

ii) If you study hard, you will do well in this exam. You will pass the course if you do well in this exam. Therefore, if you study hard, you will pass.

Hypothetical syllogism

iii) You will find this exam very hard unless you read the book. But this exam is not so hard for you. Therefore, you read the book.

Modus tollens

(b) Use rules of inference to show that the following argument form is valid. Show all steps for credit.

$$\begin{array}{l} r \lor s \\ \neg p \land q \\ r \to p \\ s \to t \\ \hline \therefore\ t \land q \end{array}$$

[1] $r \lor s$; *Hypothesis*
[2] $\neg p \land q$; *Hypothesis*
[3] $r \to p$; *Hypothesis*
[4] $s \to t$; *Hypothesis*
[5] $\neg p$; *Simplification [2]*
[6] $\neg r$;*Modus Tollens [5] & [3]*
[7] s ; *Disjunctive syllogism [6] & [1]*
[8] t ; *Modus ponens[7] & [4]*
[9] q ; *Simplification [2]*
[10] $t \land q$; *Conjunction [8] & [9]*

Question 7. [20 points, 5 each]

Determine the truth value of each of the following statements where the domain of all variables is **R** (the real numbers), and briefly justify your answers.

(a) $\forall x\ \exists y\ [\ (x^2 \geq y)\ \wedge\ (y > x)\ ]$

False. $x = 0$. $x^2 = 0$. $y = 0$. $y = x$.

(b) $\forall x\ \exists y\ [\ (x < y) \rightarrow (x \geq y + 1)\ ]$

True. For $y = x$.

(c) $\exists x\ \forall y\ [\ (y \neq 0) \rightarrow (x \cdot y = 1)\ ]$

False. For $y = 2$ *if* $x = 1$ *and* $y = 1$ *otherwise.*

(d) $\forall x\ \exists y\ \forall z\ [\ (x \neq 0) \rightarrow (\ (x \cdot y = 1) \wedge (\ (x \cdot z = 1) \rightarrow (z = y)\ )\ )\ ]$

True. For $y = 1/x$ *if* $x \neq 1$ *and* $y = 1$ *otherwise.*

Question 8. [10 points]

Prove the following statement:

If x^3 is irrational, then x is irrational.

A proof by contraposition:

If x is rational, then x = p/q for some integers p and q with q ≠ 0.
Then $x^3 = p^3/q^3$, clearly p^3 is an integer and q^3 is an integer.
We have expressed x^3 as the quotient of two integers, the second of which is not zero. This by definition means that x^3 is rational, and that completes the proof of the contrapositive of the original statement.